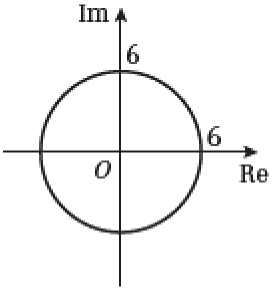
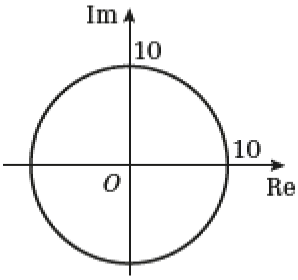
Exercise 4A

1 a $|z| = 6$



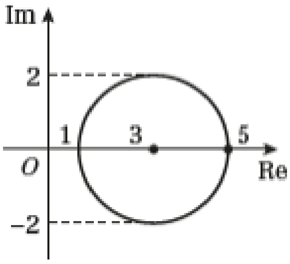
$x^2 + y^2 = 36$

b $|z| = 10$



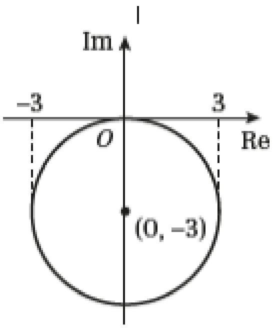
$x^2 + y^2 = 100$

c $|z - 3| = 2$



$(x - 3)^2 + y^2 = 4$

d $|z + 3\text{i}| = 3$



$x^2 + (y + 3)^2 = 9$

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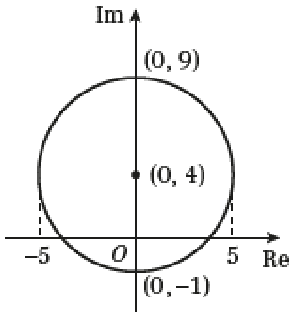
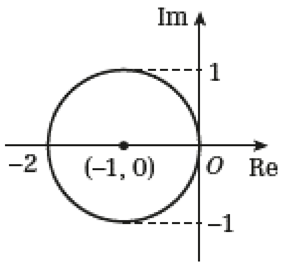
1 e $|z-4\text{i}|=5$



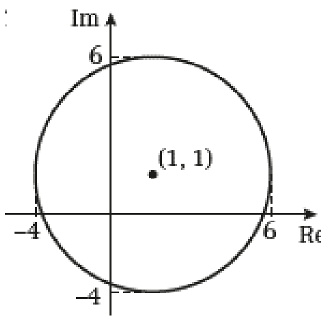
$x^2+(y-4)^2=25$

f $|z+1|=1$



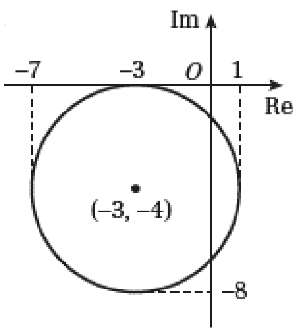
$(x+1)^2+y^2=1$

g $|z-1-\text{i}|=5$



$(x-1)^2+(y-1)^2=25$

h $|z+3+4\text{i}|=4$



$(x+3)^2+(y+4)^2=16$

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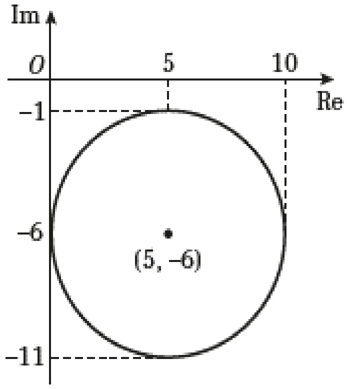
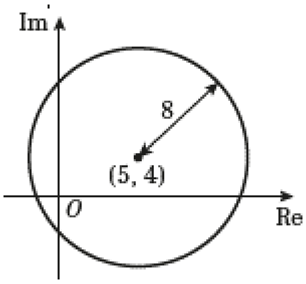
1 i $|z-5+6\text{i}|=5$



$(x-5)^2+(y+6)^2=25$

2 a $|z-5-4\text{i}|=8$



b i $|z-5-4\text{i}|=8 \Rightarrow \sqrt{(x-5)^2+(y-4)^2}=8$

$(x-5)^2+(y-4)^2=64$

When $x=0$:

$(0-5)^2+(y-4)^2=64$

$25+(y-4)^2=64$

$(y-4)^2=39$

$y=4\pm\sqrt{39}$

Therefore:

$z=\left(4\pm\sqrt{39}\right)\text{i}$

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2 b ii Re(z) = 0

When $y = 0$:

$(x-5)^2 + (0-4)^2 = 64$

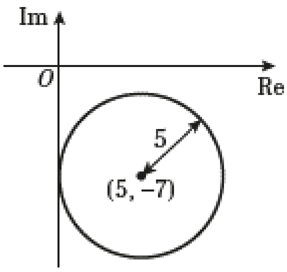
$(x-5)^2 + 16 = 64$

$(x-5)^2 = 48$

$x = 5 \pm 4\sqrt{3}$

Therefore, $z = 5 \pm 4\sqrt{3}$

3 a $|z - 5 + 7\text{i}| = 5$



b $|z - 5 + 7\text{i}| = 5 \Rightarrow \sqrt{(x-5)^2 + (y+7)^2} = 5$

$(x-5)^2 + (y+7)^2 = 25$

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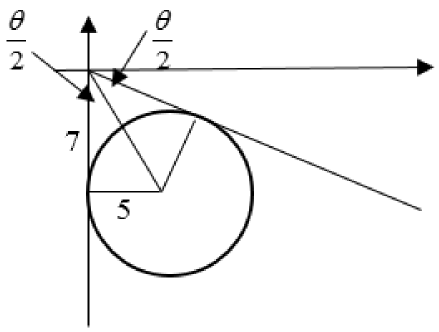
3 c



$$\tan\left(\frac{\theta}{2}\right)=\frac{5}{7}$$
$$\frac{\theta}{2}=\tan^{-1}\left(\frac{5}{7}\right)$$
$$\theta=2\tan^{-1}\left(\frac{5}{7}\right)$$

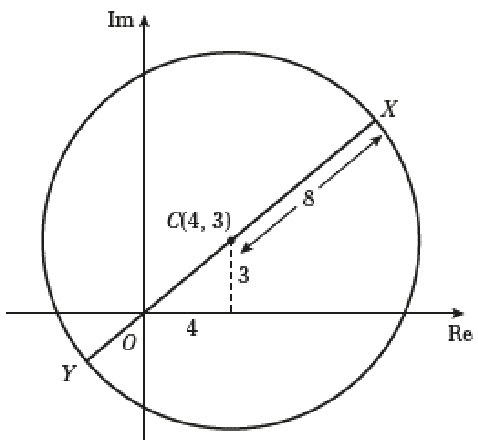
Therefore the maximum value of arg(z) is:

$$2\tan^{-1}\left(\frac{5}{7}\right)-\frac{\pi}{2}$$

4 a $|z-4-3\text{i}|=8\Rightarrow\sqrt{(x-4)^2+(y-3)^2}=8$

$$(x-4)^2+(y-3)^2=64$$

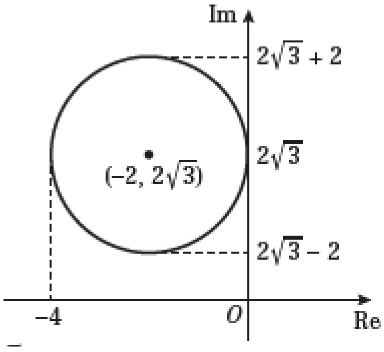
b



c $|z|_{\text{min}}=CY-CO$
$=8-5$
$=3$

$|z|_{\text{max}}=CO+CX$
$=5+8$
$=13$

5 a $\left|z+2-2\sqrt{3}\text{i}\right|=2$



b The minimum value of arg z is $\frac{\pi}{2}$

c $OC=\sqrt{(-2)^2+(2\sqrt{3})^2}=4$

$\sin\left(\frac{\theta}{2}\right)=\frac{2}{4}=\frac{1}{2}$

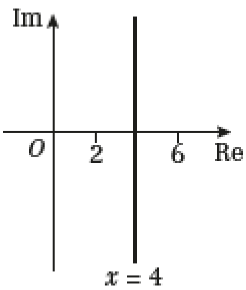
$\frac{\theta}{2}=\frac{\pi}{6}$

$\theta=\frac{\pi}{3}$

Max value of arg $z=\frac{\pi}{2}+\frac{\pi}{3}=\frac{5\pi}{6}=2.51$ rad

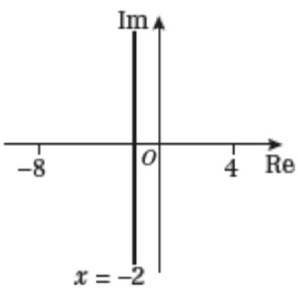
6 a $\left|z-6\right|=\left|z-2\right|$

The locus is the line $x=4$



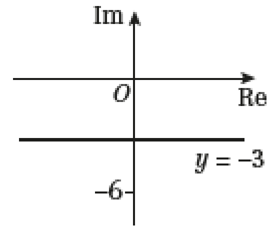
b $\left|z+8\right|=\left|z-4\right|$

The locus is the line $x=-2$



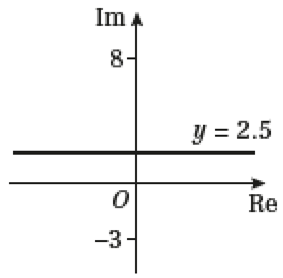
6 c $|z| = |z + 6\text{i}|$

The locus is the line $y = -3$



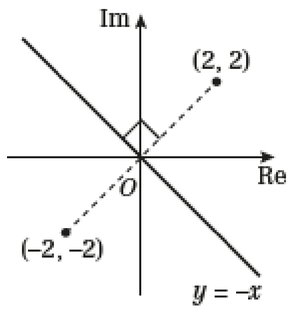
d $|z + 3\text{i}| = |z - 8\text{i}|$

The locus is the line $y = 2.5$



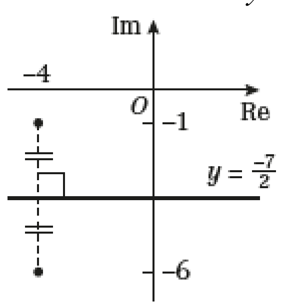
e $|z - 2 - 2\text{i}| = |z + 2 + 2\text{i}|$

The locus is the perpendicular bisector of the line segment joining
(2, 2) and (−2, −2)
The locus if the line $y = x$



f $|z + 4 + \text{i}| = |z + 4 + 6\text{i}|$

The locus is the line $y = -3.5$



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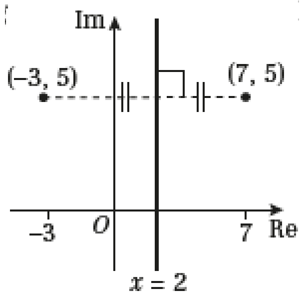
6 g $|z+3-5\text{i}| = |z-7-5\text{i}|$

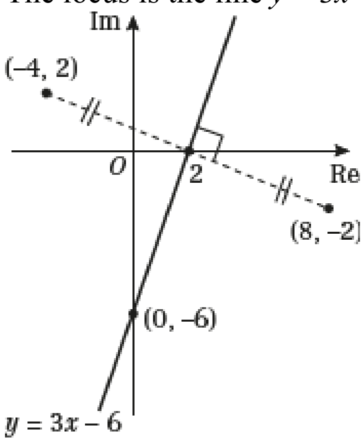
The locus is the line $x = 2$



h $|z+4-2\text{i}| = |z-8+2\text{i}|$

The locus is the perpendicular bisector of the line segment joining $(-4, 2)$ and $(8, -2)$

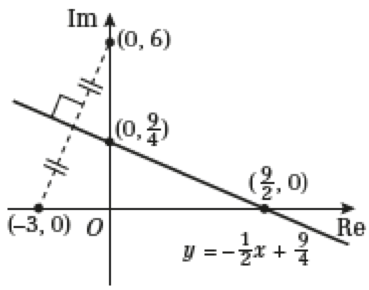
The locus is the line $y = 3x - 6$



i $\dfrac{|z+3|}{|z-6\text{i}|} = 1$

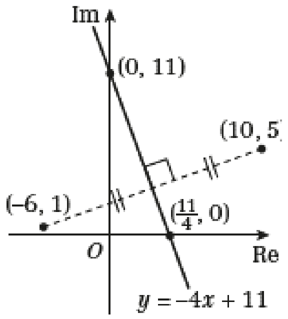
The locus is the perpendicular bisector of the line segment joining $(-3, 0)$ and $(0, 6)$

The locus is the line $y = -\frac{1}{2}x + \frac{9}{4}$

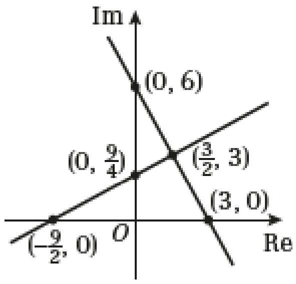


6 j $\frac{|z+6-\mathrm{i}|}{|z-10-5\mathrm{i}|}=1$ is the perpendicular bisector of the line segment joining $(-6, 1)$ and $(10,5)$

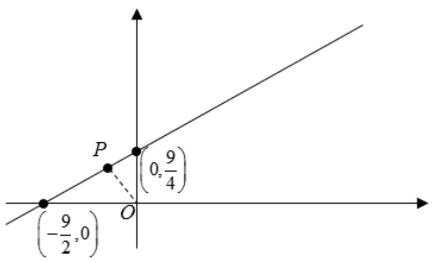
The locus is the line $y = -4x + 11$



7 a $|z-3|=|z-6\mathrm{i}|$ is the perpendicular bisector of the line segment joining $(3, 0)$ and $(0, 6)$



b



The line joining $(3, 0)$ and $(0, 6)$ has gradient -2. The gradient of the locus is $\frac{1}{2}$

Therefore the equation of the locus is: $y=\frac{1}{2}x+\frac{9}{4}$

The least possible value of $|z|$ is the magnitude of the line OP.

Since OP is perpendicular to the locus it has gradient -2.

Using $y-y_1=m(x-x_1)$ at $(0, 0)$ with $m = -2$ gives:

$y=-2x$

To find the coordinates of the point P, equate the equations of the locus and OP:

$$\frac{1}{2}x+\frac{9}{4}=-2x$$

$$\frac{5}{2}x=-\frac{9}{4} \Rightarrow x=-\frac{9}{10} \text{ and } y=\frac{18}{10}$$

$$|OP|=\sqrt{\left(-\frac{9}{10}\right)^2+\left(\frac{18}{10}\right)^2}$$

$$=\frac{9\sqrt{5}}{10}$$

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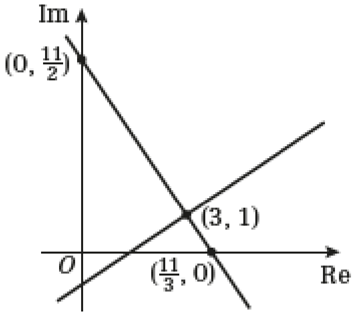
8 a $|z+3+3\text{i}| = |z-9-5\text{i}|$ is the perpendicular bisector of the line segment joining (−3, −3) and (9, 5)



b The gradient of the line joining (−3, −3) and (9, 5) is $m = \frac{5+3}{9+3} = \frac{2}{3}$

The gradient of the locus is, therefore, $-\frac{3}{2}$

The midpoint of the line joining (−3, −3) and (9, 5) is (3, 1)

Using $y - y_1 = m(x - x_1)$ at (3, 1) with $m = -\frac{3}{2}$ gives:

$$y - 1 = -\frac{3}{2}(x-3)$$

$$y - 1 = -\frac{3}{2}x + \frac{9}{2}$$

$$y = -\frac{3}{2}x + \frac{11}{2}$$

c The gradient of the perpendicular from the locus is $\frac{2}{3}$

Using $y - y_1 = m(x - x_1)$ at (0, 0) with $m = \frac{2}{3}$ gives:

$$y = \frac{2}{3}x$$

To find the coordinates of the point, P, where the perpendicular and the locus meet equate the equation of the locus and the equation of OP.

$$-\frac{3}{2}x + \frac{11}{2} = \frac{2}{3}x$$

$$\frac{13}{6}x = \frac{11}{2}$$

$$x = \frac{33}{13} \text{ and } x = \frac{22}{13}$$

The least possible value of $|z|$ is the magnitude of the line OP.

$$|OP| = \sqrt{\left(\frac{33}{13}\right)^2 + \left(\frac{22}{13}\right)^2}$$

$$= \frac{11\sqrt{13}}{13}$$

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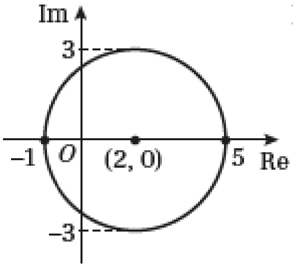
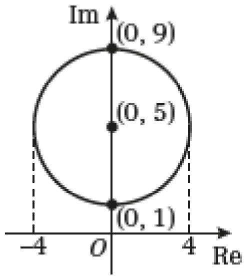
9 a $|2-z|=3$ is the circle with centre (2, 0) and radius 3



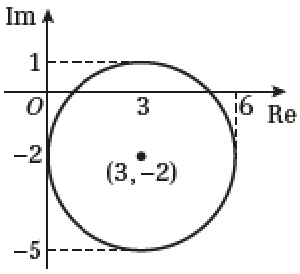
$(x-2)^2+y^2=9$

b $|5\text{i}-z|=4$ is the circle with centre (0, 5) and radius 4



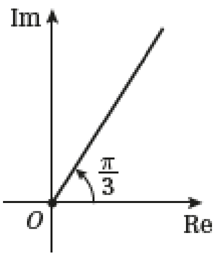
$x^2+(y-5)^2=16$

c $|3-2\text{i}-z|=3$ is the circle with centre (3, −2) and radius 3

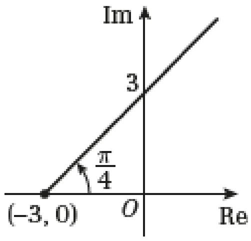


$(x-3)^2+(y+2)^2=9$

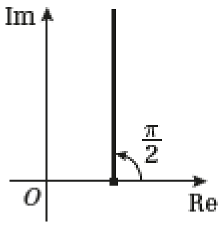
10 a $\arg z=\frac{\pi}{3}$ is the half-line originating at (0, 0) at an angle of $\frac{\pi}{3}$ to the positive x-axis.



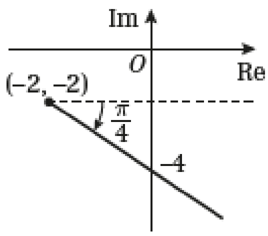
10 b $\arg(z+3)=\frac{\pi}{4}$ is the half-line originating at (−3, 0) at an angle of $\frac{\pi}{4}$ to the positive x-axis.



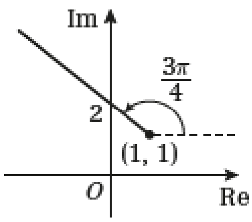
c $\arg(z-2)=\frac{\pi}{2}$ is the half-line originating at (2, 0) at an angle of $\frac{\pi}{2}$ to the positive x-axis.



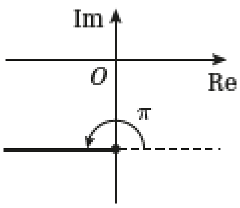
d $\arg(z+2+2\text{i})=-\frac{\pi}{4}$ is the half-line originating at (−2, −2) at an angle of $-\frac{\pi}{4}$ to the positive x-axis.



e $\arg(z-1-\text{i})=\frac{3\pi}{4}$ is the half-line originating at (1, 1) at an angle of $\frac{3\pi}{4}$ to the positive x-axis.



f $\arg(z+3\text{i})=\pi$ is the half-line originating at (0, −3) at an angle of π to the positive x-axis.



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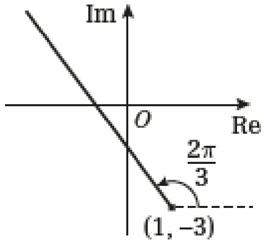
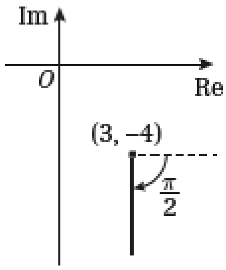
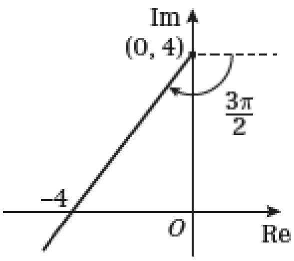
10 g $\arg(z-1+3\text{i})=\frac{2\pi}{3}$ is the half-line originating at (1, −3) at an angle of $\frac{2\pi}{3}$ to the positive x-axis.



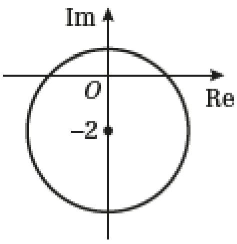
h $\arg(z-3+4\text{i})=-\frac{\pi}{2}$ is the half-line originating at (3, −4) at an angle of $-\frac{\pi}{2}$ to the positive x-axis.



i $\arg(z-4\text{i})=-\frac{3\pi}{2}$ is the half-line originating at (0, 4) at an angle of $-\frac{3\pi}{2}$ to the positive x-axis



11 a $|z+2\text{i}|=3$ is the circle with centre (0, −2) and radius 3.



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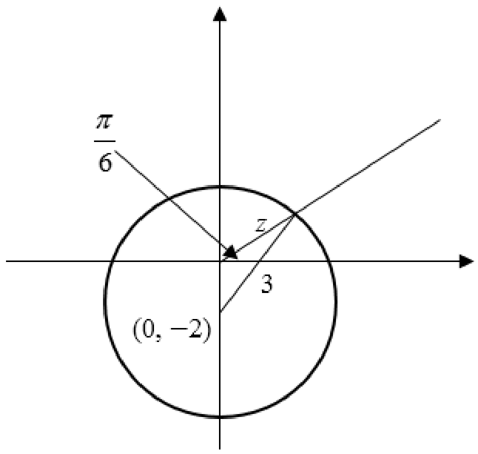
11 b



$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\left(\frac{2\pi}{3}\right) = \frac{|z|^2 + 2^2 - 3^2}{4|z|}$$

$$-2|z| = |z|^2 - 5$$

$$|z|^2 + 2|z| - 5 = 0$$

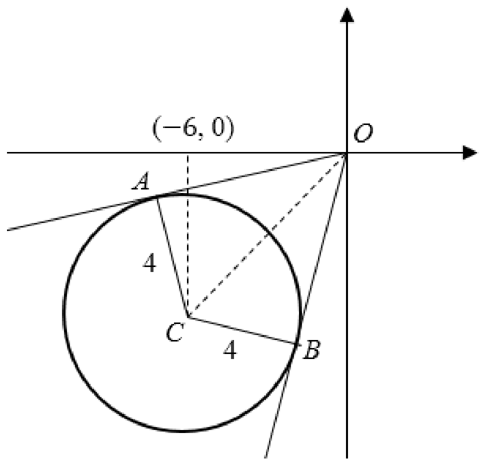
$$|z| = \frac{-2 \pm \sqrt{2^2 - 4(1)(-5)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{24}}{2}$$

$$= -1 \pm \sqrt{6}$$

Since $|z|$ is positive $|z| = -1 + \sqrt{6}$

12 a $|z + 6 + 6\mathrm{i}| = 4$ is the circle with centre $(-6, -6)$ and radius 4



$$|OC| = \sqrt{6^2 + 6^2}$$

$$= 6\sqrt{2}$$

Therefore $|z_{\max}| = 6\sqrt{2} + 4$ and $|z_{\min}| = 6\sqrt{2} - 4$

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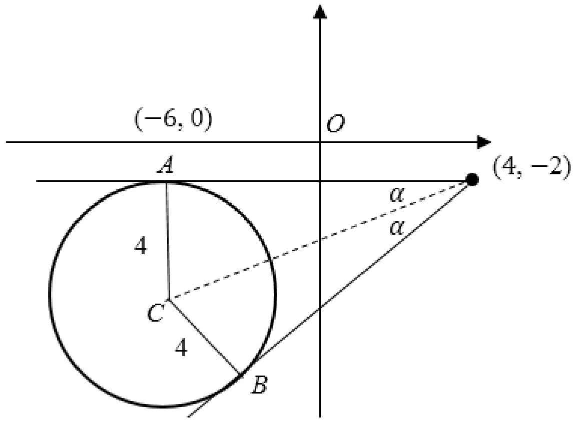
12 b



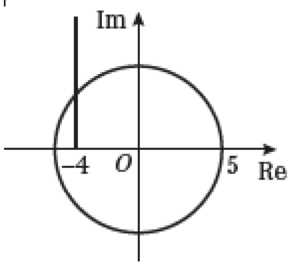
$\tan \alpha = \frac{4}{10}$

$\alpha = 0.3805...$ therefore $2\alpha = 0.7610...$

$-\pi + 2\alpha = -2.3805...$

Therefore, range of values with no common solutions is $(-2.38, \pi)$

13 a $|z| = 5$ is the circle with centre (0, 0) and radius 5



b The equation of the circle is:

$x^2 + y^2 = 25$

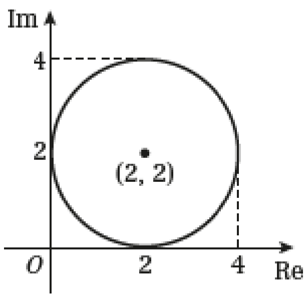
When $x = -4$

$16 + y^2 = 25$

$y = \pm 3$

Since the complex number must satisy both $|z| = 5$ and $\arg(z+4) = \frac{\pi}{2}$ it is $z = -4 + 3\text{i}$

14 a $|z - 2 - 2\text{i}| = 2$ is the circle with centre (2, 2) and radius 2



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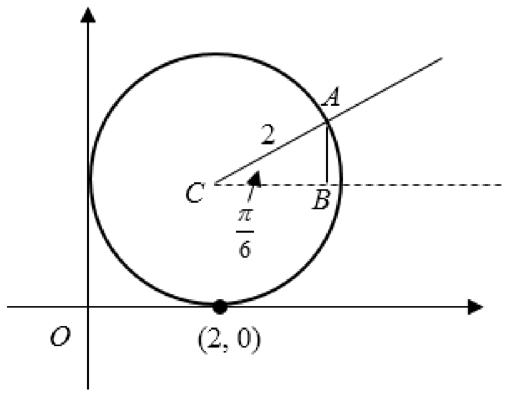
14 b $\arg(z-2-2\mathrm{i})=\dfrac{\pi}{6}$



$\sin\left(\dfrac{\pi}{6}\right)=\dfrac{AB}{2}$

$AB=2\sin\left(\dfrac{\pi}{6}\right)$

$=1$

So the perpendicular distance from the x-axis to A is 3

$\cos\left(\dfrac{\pi}{6}\right)=\dfrac{CB}{2}$

$CB=2\cos\left(\dfrac{\pi}{6}\right)$

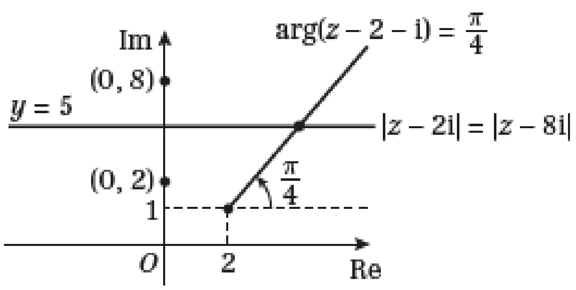
$=\sqrt{3}$

So the perpendicular distance from the y-axis to A is $2+\sqrt{3}$

So $z=\left(2+\sqrt{3}\right)+3\mathrm{i}$

15 a $|z-2\mathrm{i}|=|z-8\mathrm{i}|$ is the line $y=5$

b $\arg(z-2-\mathrm{i})=\dfrac{\pi}{4}$ is the half-line originating from (2, 1) at $\dfrac{\pi}{4}$ to the positive horizontal axis.



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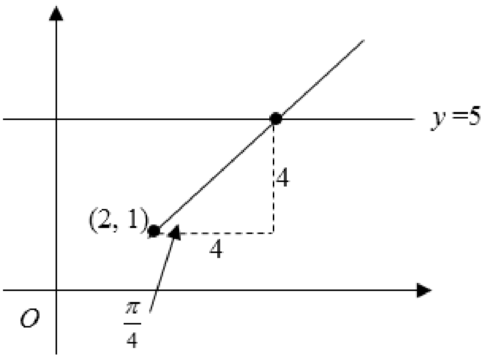
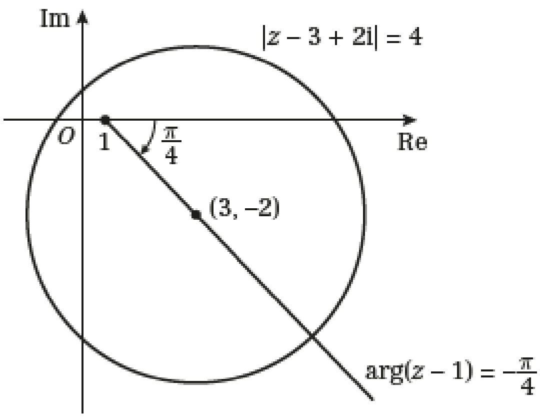
15 c



$z = 6 + 5i$

16 a $|z-3+2i| = 4$ is the circle with centre (3, −2) and radius 4

b $\arg(z-1) = -\frac{\pi}{4}$ is the half-line originating from (1, 0) at $-\frac{\pi}{4}$ to the positive horizontal axis.



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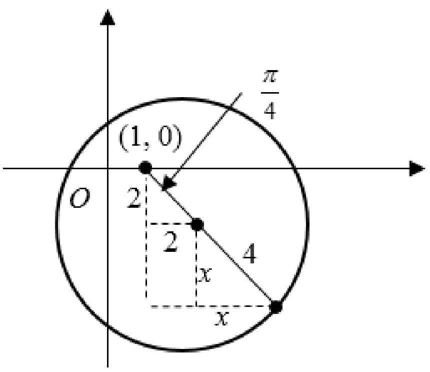

$2x^2 = 16$

$x = 2\sqrt{2}$

Therefore:

$a = 3 + 2\sqrt{2}$

$b = -2 - 2\sqrt{2}$

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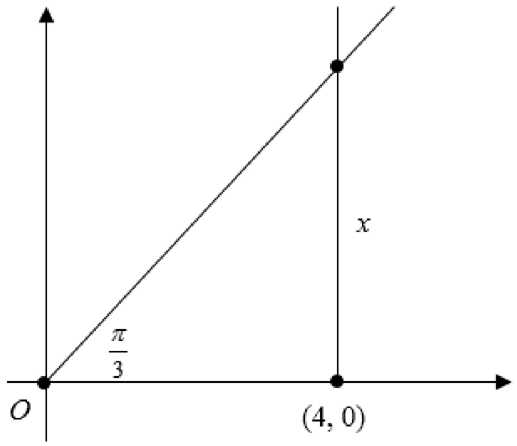
17 a $\arg(z)=\frac{\pi}{3}$ is the half-line originating from (0, 0) at $\frac{\pi}{3}$ to the positive horizontal axis.

$\arg(z-4)=\frac{\pi}{2}$ is the half-line originating from (4, 0) at $\frac{\pi}{2}$ to the positive horizontal axis.



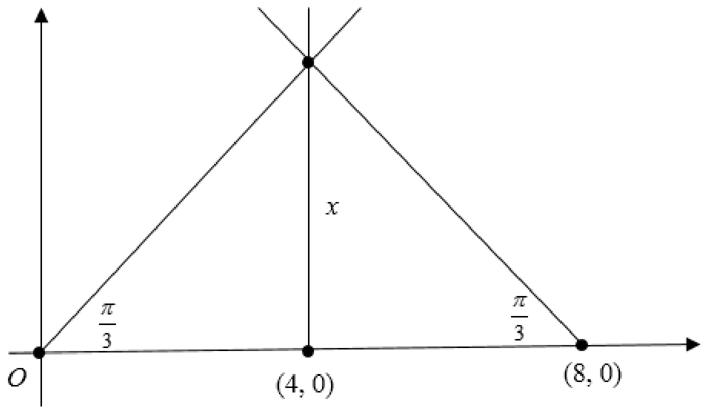
$\tan\left(\frac{\pi}{3}\right)=\frac{x}{4}$

$x=4\tan\left(\frac{\pi}{3}\right)$

$=4\sqrt{3}$

Therefore $z=4+4\sqrt{3}\mathrm{i}$

b



$\arg(z-8)$ is the angle between the positive x-axis and the half-line

Therefore, $\arg(z-8)=\frac{2\pi}{3}$

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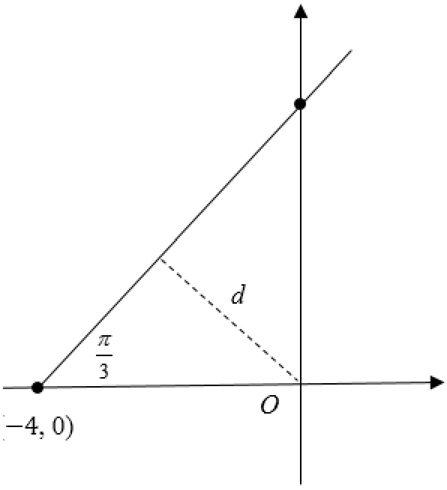
18 a $\arg(z+4)=\frac{\pi}{3}$ is the half-line originating from (−4, 0) at $\frac{\pi}{3}$ to the positive horizontal axis.



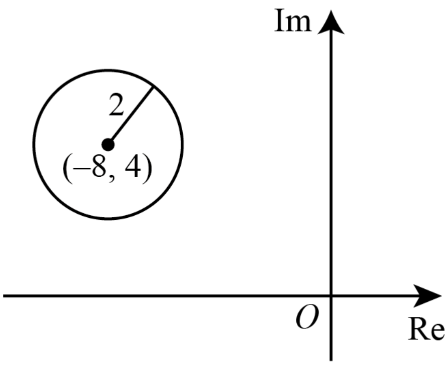
b $\sin\left(\frac{\pi}{3}\right)=\frac{d}{4}$

$d=4\sin\left(\frac{\pi}{3}\right)$

$=2\sqrt{3}$

Therefore $|z_{\min}| = 2\sqrt{3}$

19 a $|z+8-4\text{i}|=2$ is the circle with centre (−8, 4) and radius 2



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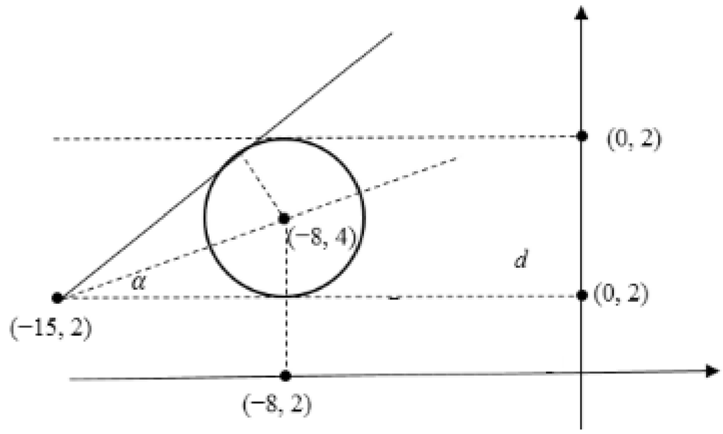
19 b $\arg(z+15-2i)=\frac{\pi}{3}$ is the half-line originating from (−15, 2) at 2α to the positive horizontal axis.



$\tan\alpha=\frac{2}{7}\Rightarrow\sin\alpha=\frac{2}{\sqrt{53}}$

$\alpha=\sin^{-1}\left(\frac{2}{\sqrt{53}}\right)$

$2\alpha=2\sin^{-1}\left(\frac{2}{\sqrt{53}}\right)$ as required

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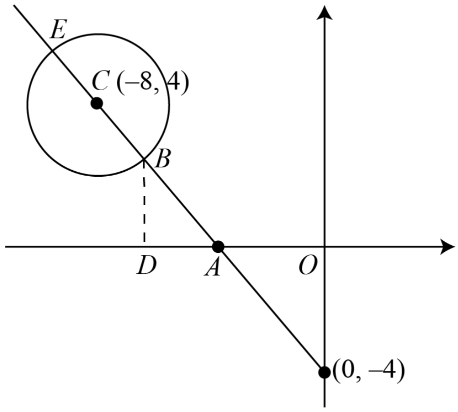
19 c



$|AC| = \sqrt{4^2 + 4^2}$

$= 4\sqrt{2}$

Therefore:

$|BA| = 4\sqrt{2} - 2$

$\cos\left(\frac{\pi}{4}\right) = \frac{|AD|}{4\sqrt{2} - 2}$

$AD = 4 - \sqrt{2}$

So the x-coordinate is:

$-4 - \left(4 - \sqrt{2}\right) = -8 + \sqrt{2}$

And the y-coordinate is:

$4 - \sqrt{2}$

The coordinates of B are $\left(-8 + \sqrt{2}, 4 - \sqrt{2}\right)$

And by symmetry the coordinates of E are $\left(-8 - \sqrt{2}, 4 + \sqrt{2}\right)$

z = $(-8 + \sqrt{2}) + (4 - \sqrt{2})$I and $z = (-8 - \sqrt{2}) + (4 + \sqrt{2})$i

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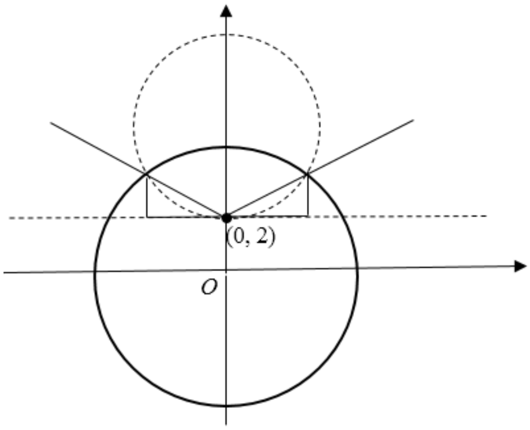
Challenge

$|z+\text{i}|=5$ is the circle with centre (0, −1) and radius 5

$|z-4\text{i}|<3$ is area enclosed by the circle with centre (0, 4) and radius < 3

$\arg(z-2\text{i})=\theta,\ -\pi<\theta\le\pi$



The Cartesian equations on the circles are:

$x^2+(y+1)^2=25$ and $x^2+(y-4)^2=9$

$25-(y+1)^2+(y-4)^2=9$

$25-(y^2+2y+1)+(y^2-8y+16)=9$

$40-10y=9$

$10y=31$

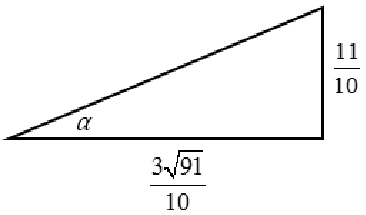
$y=\frac{31}{10}$

Therefore:

$x^2+\left(\frac{41}{10}\right)^2=25$

$x^2=\frac{819}{100}$

$x^2=\pm\frac{3\sqrt{91}}{10}$



$\tan\alpha=\frac{\frac{11}{10}}{\frac{3\sqrt{91}}{10}}$

$=0.3843...$

$\pi-\alpha=2.7572...$

$0.384<\theta<2.76$(3 s.f.)

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